Temperature behavior of vortices of a 3D thermoconducting viscous fluid

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Abstract

The Navier-Stokes-Fourier model for a 3D thermoconducting viscous fluid, where the evolution equation for the temperature T contains a term proportional to the rate of energy dissipation, is investigated analitically at the light of the rotational invariance property. Two cases are considered: the Couette flow and a flow with a radial velocity between two rotating impermeable and porous coaxial cylinders, respectively. In both cases, we show the existence of a maximum value of T, $T_{\max}$, when the difference of temperature $\Delta T = T_2 - T_1$ on the surfaces of the cylinders is assigned. The role of $T_{\max}$ is discussed in the context of different physical situations.

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In Ref. [1], a thermoconducting incompressible viscous fluid system, named Navier-Stokes-Fourier (NSF) model, is presented. In 3D, this model is governed by the equations

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\left(\frac{p}{\rho}\right) - \nu\nabla^2\mathbf{u} = 0, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2}$$

$$T_t + \mathbf{u}\cdot\nabla T - k_H\nabla^2 T = \frac{\eta}{\rho C_p}\sum_{i,j}(\partial_i u_j + \partial_j u_i)^2, \tag{3}$$

where $\mathbf{u}$ is the velocity field, $T = T(x,y,z,t)$ is a passive scalar identified by the fluid temperature [2], $p = p(x,y,z,t)$ the fluid pressure, ρ the fluid density, η the dynamic viscosity,

$\nu = \eta/\rho$ the kinematic viscosity, C_p the heat capacity, κ the heat conductivity, $k_H = \kappa/(\rho C_p)$ the thermal diffusivity, and $\partial_i \equiv \partial/\partial x_i$ ($x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$).

The r.h.s. in (3) is related to the rate of energy dissipation $\varepsilon = 2\nu S^2$, where $S^2 = S_{ij}S_{ij}$ and $S_{ij} = (1/2)(\partial_i u_j + \partial_j u_i)$ are the strain matrix elements. Among the many questions arising in the study of the model (1)-(3), such as for instance the investigation of statistical solutions and the onset of turbulence [3], the presence of a dissipation term in the equation for the temperature deserves a special attention. At the best of our knowledge, in three spatial dimensions Eqs. (1)-(3) have never been studied by exploiting an analytic procedure. In this Letter we discuss some new thermal effects described by exact solutions of the vortex type of the NSF model. In particular, we have analyzed a steady Taylor-Couette flow for an incompressible fluid entrapped between two rotating coaxial impermeable cylinders, and a flow with an additional radial component of the velocity, when the walls of the cylinders are porous (for technical details on the experimental devices, see [4,5]). As is well-known, the hydrodynamic problems connected with the system of two coaxial rotating cylinders has been intensively studied from an experimental and theoretical point of view since the work of Couette (1890) [6]. The main motivation was the discovery by Taylor of unstable flow regimes consisting in toroidal vortices (1923), which appear for high values of the Taylor number [7]. This number can be defined in several forms. Following [5] we will use $Ta = r_1 \Omega_1 d/\nu$, where r_1 is the radius of the inner cylinder, d the width of the gap between the two cylinders, Ω_1 the rotational velocity of the inner cylinder. In our context, we will consider low Taylor numbers ($Ta \sim 100 \div 200$), in order to preserve the geometrical symmetries of the model.

In the two cases of impermeable and porous cylinders, we show the existence of a value $T_{\max}$ for the temperature field associated with the fluid, when the difference of temperature $\Delta T = T_2 - T_1$ on the surfaces of the cylinders is assigned for asymptotic values of time. This maximum appears as a consequence of a mechanism of energy dissipation in heat, due to viscous effects, and depends on the width of the gap between the cylinders, on their relative angular velocity, and on the fixed value of ΔT. Remarkably, we found that the position of $T_{\max}$ as a function of the radial coordinate r can vary continuously in the gap $r_1 \leq r \leq r_2$

between the two cylinders by varying the parameter ΔT.

In the NSF model, ρ is taken to be constant. Although this assumption could appear as a too severe restriction on the possible modelling of realistic situations, nevertheless the NSF model deserves to be analyzed as a guide-framework for the investigation of less ideal systems, such as the Rayleigh and the Lorentz models, which describe incompressible fluid motions where thermal phenomena are more significant [1].

We found that the NSF model allows Lie-point symmetries generated by infinitesimal operators giving finite group transformations and exact solutions via the corresponding reduced equations. (The technical machinery yielding these reductions is outlined in [8,9]). Many of these symmetries have a nice geometric interpretation: they express the invariance of the NSF system under rotations, translations, Galilean boosts and scale transformations. Therefore, as one expects, their expressions do not explicitly contain the viscosity ν. Nevertheless, such symmetries hold only in the viscous case, in the sense that the corresponding symmetries for the inviscid situation cannot be reproduced by those for $\nu \neq 0$ by setting $\nu = 0$.

Here we shall limit ourselves to deal with the rotational symmetry defined by the generator

$$V_R = y\partial_x - x\partial_y + u_2\partial_{u_1} - u_1\partial_{u_2}. \tag{4}$$

A more systematic treatment of the symmetry properties of the NSF model together with the investigation of the role of the boundary conditions will be reported in a separated paper. The operator (4) gives rise to a reduced NSF (RNSF) system whose last equation is

$$\begin{aligned}&\theta_t + U_1\theta_r + U_3\theta_z - \frac{\eta}{\rho C_p}\left(2U_{1r}^2 + 2\frac{U_1^2}{r^2} + U_{1z}^2 + U_{2r}^2 - \right.\\ &\left. 2\frac{U_2U_{2r}}{r} + \frac{U_2^2}{r^2} + U_{2z}^2 + U_{3r}^2 + 2U_{3z}^2 + 2U_{1z}U_{3r}\right) - \\ &k_H\left(\theta_{rr} + \frac{\theta_r}{r} + \theta_{zz}\right) = 0,\end{aligned} \tag{5}$$

where z, t, $r = \sqrt{x^2+y^2}$ are independent symmetry variables and U_1, U_2, U_3, Π, θ, related to the original variables by

$$U_1 = u_1 \cos\varphi + u_2 \sin\varphi,\ U_2 = -u_1 \sin\varphi + u_2 \cos\varphi,$$

$$U_3 = u_3, p = \Pi, T = \theta, \tag{6}$$

are dependent variables expressed in terms of z, t, r. Here $\cos\varphi = x/r$, $\sin\varphi = y/r$, U_1, U_2 can be interpreted as the radial and the azimuthal components of the (reduced) velocity $\mathbf{U}$, and U_3 is the component along the z-axis.

We find that the RNSF system admits two solutions which are, respctively, *i)* the circular Couette flow (in which only the azimuthal component U_2 of $\mathbf{U}$ is different from zero), and *ii)* a flow where all the three components of $\mathbf{U}$ are different from zero. In this case U_3 turns out to be time dependent in such a way that $U_3 \to 0$ exponentially when the adimensional variable $\mu_0 t$ goes to infinity (see (14)). In the original velocity components u_1, u_2 and u_3 the circular character of the Couette flow appears, while the flow of case *ii)* shows a spiral behavior.

Case i)

A simple but notable solution to the RNSF system is obtained by choosing $U_1 = U_3 = 0$ and

$$U_2 = \frac{\alpha}{r} + \beta r, \tag{7}$$

where α, β are constants. Then

$$p = -\frac{\alpha^2 \rho}{2r^2} + \frac{1}{2}\beta^2 \rho r^2 + 2\alpha\beta\rho \ln r, \tag{8}$$

apart from an arbitrary function of time, while the temperature T is provided by

$$T = \exp(-\lambda_0 t) \times \left[c_1 J_0\left(\sqrt{\frac{\lambda_0 - \lambda_1}{k_H}} r\right) + c_2 Y_0\left(\sqrt{\frac{\lambda_0 - \lambda_1}{k_H}} r\right)\right] \times \left[c_3 \cos\sqrt{\frac{\lambda_1}{k_H}} z + c_4 \sin\sqrt{\frac{\lambda_1}{k_H}} z\right] + T_d^I, \tag{9}$$

where

$$T_d^I = \lambda_2 \ln r - \frac{\alpha^2 \eta}{k_H \rho C_p} \frac{1}{r^2} + T_0^I \tag{10}$$

is the contribution to the temperature coming from the energy dissipation present in (3) and $\lambda_0 > \lambda_1 > 0$, λ_2, T_0^I and c_j are constants. Here J_0 and Y_0 are the Bessel functions of the first and the second kind, respectively (see [10], p. 358).

The original velocity components corresponding to the reduced velocity $\mathbf{U} \equiv (0, \frac{\alpha}{r} + \beta r, 0)$ are

$$u_1 = -\left(\frac{\alpha}{r^2} + \beta\right) y, \; u_2 = \left(\frac{\alpha}{r^2} + \beta\right) x, \; u_3 = 0, \tag{11}$$

from which we have $r = r_0 = const$. Equations (11) are solved by $x = r_0 \cos Kt$, $y = r_0 \sin Kt$, $z = const$ where $K = \alpha / r_0^2 + \beta$.

The reduced solution (7) is pertinent to a steady flow where the fluid occupies the gap $r_1 \leq r \leq r_2$ between two coaxial cylinders of radii r_1 and r_2 rotating with (constant) angular velocities Ω_1 and Ω_2 [11]. Under the hypothesis of no-slip condition on the cylinders, α and β take the form $\alpha = [r_1^2 r_2^2 (\Omega_2 - \Omega_1)]/(r_1^2 - r_2^2)$, $\beta = (r_1^2 \Omega_1 - r_2^2 \Omega_2)/(r_1^2 - r_2^2)$, while the vorticity $\boldsymbol{\omega_R}$ associated with the (reduced) velocity (7) coincides with the vorticity $\boldsymbol{\omega}$ for the velocity (11).

In Fig. 1 we show the behavior of the temperature field, as being described by the solution (9), as a function of t and r (brighter regions correspond to higher values of T). By exploiting the scale invariance of the NSF system we can express the physical constants in an adimensional form [12]. Hereafter we will assume $\nu = 10^{-2}$, $\rho = 1$, $\eta = 10^{-2}$, $C_p = 1$, $\kappa = 6 \times 10^{-3}$. After a transient phase, only the contribution (10), due to the dissipative term in (3) survives. In this situation we have a maximum of T for any t in the gap between the two cylinders. This maximum, as shown in Fig. 2 for the asymptotic case, can continuously vary by varying ΔT (or equivalently λ_2).

FIGURES

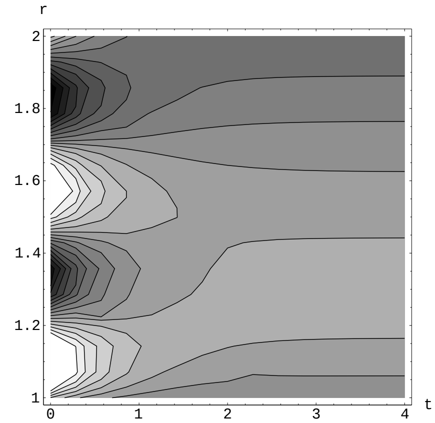

FIG. 1. Temperature field for the motion of a fluid between two coaxial rigid and impermeable cylinders with radii $r_1 = 1$ and $r_2 = 2$ for $z = 0.5$ ($\alpha = 1$).

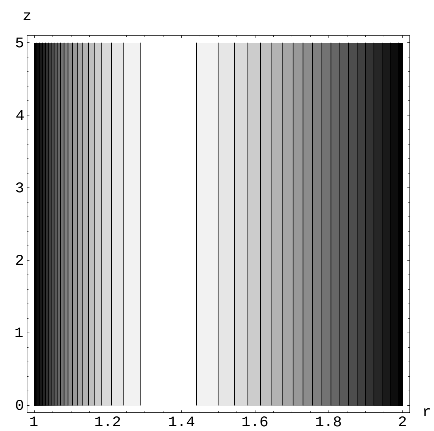

FIG. 2. Asymptotic behavior of the thermal field for $1 \leq r \leq 2$, $0 \leq z \leq 5$, when $\Delta T = 0$ ($Ta = 150$).

In Fig. 3, we plot $T - T_1$ vs. r for different values of $b = \frac{\Omega_2}{\Omega_1}$, in the case $\Delta T = 0$. We point out that when the cylinders rotate in the opposite sense ($b < 0$), $T_{\max}$ increases. This is related to a growth of the energy dissipation induced by friction effects.

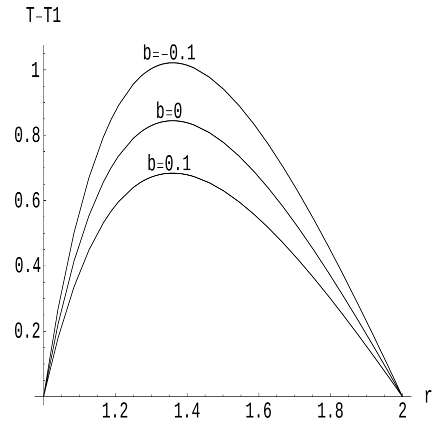

FIG. 3. Behavior of the thermal field as a function of the radial distance between the two rigid and impermeable cylinders for some values of the parameter $b = \frac{\Omega_2}{\Omega_1}$ with $r_1 = 1$ and $r_2 = 2$ $(Ta = 150)$.

Case ii)

Another interesting vortex solution of the NSF equations (1)-(3) can be found by putting

$$U_1 = \frac{\gamma\nu}{r},\ U_2 = Ar^{\gamma+1} + \frac{B}{r}, \tag{12}$$

into the RNSF system, where A, B and γ are constants.

The pressure $p \equiv \Pi$ is given by

$$p = \frac{\rho(4ABr^{\gamma+2} - \gamma B^2 - \gamma^3\nu^2)}{2\gamma r^2} + \frac{\rho A^2 r^{2(\gamma+1)}}{2(\gamma+1)}, \tag{13}$$

apart from an arbitrary function of time, while the RNSF system yields for U_3:

$$U_3 = \exp(-\sigma_0 t) r^{\frac{\gamma}{2}} \times \\ \left[A_0 J_{\frac{\gamma}{2}}\left(\sqrt{\frac{\sigma_0}{\nu}} r\right) + A_1 Y_{\frac{\gamma}{2}}\left(\sqrt{\frac{\sigma_0}{\nu}} r\right)\right], \tag{14}$$

where $\sigma_0 > 0$, A_0, A_1 are constants.

An interesting physical interpretation of the vortex solution presented in *Case ii)* is the following. Let us consider a device constituted by two rotating porous coaxial cylinders of radii $r_1 < r_2$ containing in the gap a fluid of kinematic viscosity ν.

Let $\gamma = \frac{u_0 r_1}{\nu}$ be the radial Reynolds number, and $u_0 \equiv U_1(r_1)$ the radial velocity through the wall of the inner cylinder [4]. We assume that the flow is inward for $\gamma < 0$ and outward for $\gamma > 0$. For $A_0 = A_1 = 0$, the velocity filed reproduces the solution investigated in [4]. Furthermore, for $\gamma = 0$ the quantities (12) correspond to the Couette flow (7). In the case of porous cylinders, A and B are related to the geometry and the dynamics of the device, and explicitly become: $A = -[\Omega_1 a^2(1 - b/a^2)]/[r_2^\gamma(1 - a^{\gamma+2})]$ and $B = r_1^2\Omega_1(1 - ba^\gamma)/(1 - a^{\gamma+2})$, where $a = r_1/r_2$ and $b = \Omega_2/\Omega_1$.

In general, i.e. for $U_3 \neq 0$, the evolution equation (5) for the temperature is very complicated and the finding of exact solutions is a difficult task. However, for $A_0 = A_1 = 0$, we obtain

$$\begin{aligned}
&T = \exp(-\mu_0 t) r^{\frac{\gamma\nu}{2k_H}} \times \\
&\left[c_1 J_{\frac{\gamma\nu}{2k_H}}\left(\sqrt{\frac{\mu_1}{k_H}} r\right) + c_2 Y_{\frac{\gamma\nu}{2k_H}}\left(\sqrt{\frac{\mu_1}{k_H}} r\right)\right] \times \\
&\left[c_3 \sin\left(\sqrt{\frac{\mu_0 - \mu_1}{k_H}} z\right) + c_4 \cos\left(\sqrt{\frac{\mu_0 - \mu_1}{k_H}} z\right)\right] + T_d^{II},
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
T_d^{II} &= -\frac{2\eta(B^2 + \gamma^2\nu^2)}{\rho C_p(2k_H + \gamma\nu)r^2} + \frac{4\eta AB}{\gamma\rho C_p(k_H - \nu)} r^\gamma + \\
&\mu_2 r^{\frac{\gamma\nu}{k_H}} - \frac{\gamma^2\eta A^2 r^{2(\gamma+1)}}{2\rho C_p(\gamma+1)[2k_H(\gamma+1) - \gamma\nu]} + T_0^{II}
\end{aligned} \tag{16}$$

Here $\mu_0 > \mu_1 > 0$, μ_2 and T_0^{II} are constants, the Prandtl number $\Pr = \nu/k_H$ is supposed to be $\neq 1$ and $\gamma > 0$ (i.e. the radial flow is considered outward) to avoid singularities.

To determine the time behavior of the coordinates $(x(t), y(t), z(t))$ of the fluid particle, we integrate the velocity field $\mathbf{u} = (u_1, u_2, u_3)$ to get $r^2 = 2Ct + r_0^2$, where x and y obey the pair of equations of the time-dependent oscillator type which afford the solutions $x = \sqrt{\tau}\cos\psi$, $y = \sqrt{\tau}\sin\psi$ where $\tau = 2Ct + r_0^2$ and $\psi = \frac{1}{2C}[\frac{2A}{\gamma+2}\tau^{\frac{\gamma}{2}+1} + B\ln\frac{\tau}{\tau_0}]$.

The term (16) is again due to the presence of the dissipation rate in (3), and represents the asymptotic limit of the solution (15), for $t \to \infty$. In Fig. 4 we plot the temperature field (15) for $z = 0.5$. This plot is analogous to that of Fig. (1). In other words, also in this case a maximum of T emerges after a transient phase.

The position at which $T_{\max}$ is located in the gap $r_1 \leq r \leq r_2$ (Fig. 5) depends on the value of γ, on the difference of temperature ΔT and quadratically on the width of the gap.

In particular, $T_{\max}$ tends to migrate towards the well of the outer cylinder when the values of γ increase and vice-versa. A similar situation occurs when ΔT changes.

In Figs. 6 and 7 we plot the thermal field for $b = \frac{\Omega_2}{\Omega_1} \geq 0$ and $b < 0$, when $\Delta T = 0$ and $\Delta T = 4$.

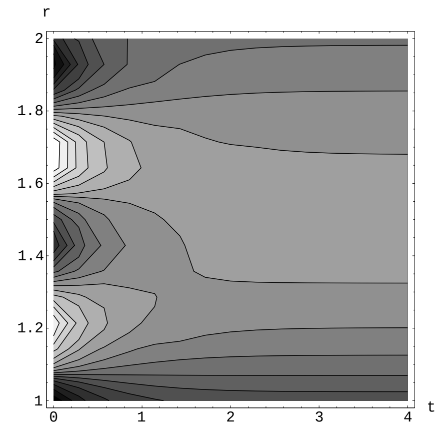

FIG. 4. As in Fig. 1 for the case of porous cylinders ($\gamma = 1$).

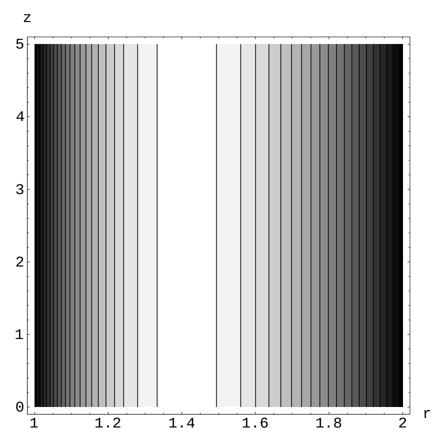

FIG. 5. Asymptotic behavior of the temperature field between two rotating porous cylinders as a function of r ($1 \leq r \leq 2$) and z ($0 \leq z \leq 5$) for $\Delta T = 0$, $\gamma = 5$ and $b = \frac{\Omega_2}{\Omega_1} = 0.2$ ($Ta = 100$).

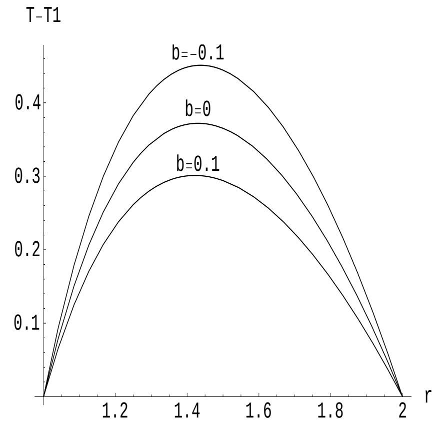

FIG. 6. Temperature behavior as a function of the radial distance between the two rotating porous cylinders for several values of $b = \frac{\Omega_2}{\Omega_1}$ and $\gamma = 1$ if the cylinders temperatures are kept equal ($Ta = 100$).

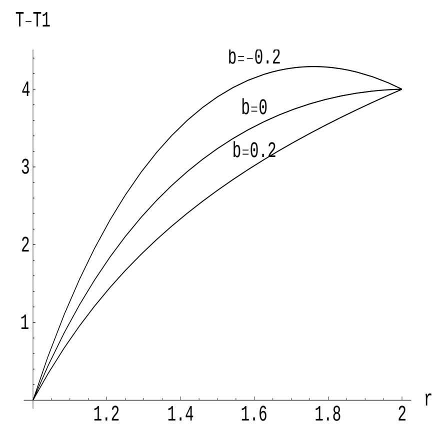

FIG. 7. As in Fig. 6, where now $\Delta T = 4$ and $\gamma = 1$ ($Ta = 200$).

To conclude, from our analysis of the temperature behavior of vortices of the NSF model (1)-(3), notable thermal effects arise as a consequence of the presence of the energy dissipation term in the evolution equation (3) for the temperature field. We hope that our theoretical results can stimulate some experimental work addressed to a possible evidence of such effects.

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